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Projective Geometry for Polarizations in Geometric Quantization

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It is important to know the extent to which the procedure of geometric quantization depends on a choice of polarization of the symplectic manifold that is the classical phase space. Published results have so far been restricted to real and transversal polarizations. It turns out that there is a natural characterization of real transversal Lagrangian distributions and maps among them using projective concepts. We give explicit constructions for R^{2n}.

1. INTRODUCTION

The geometric quantization theory of Kostant (1970) provides for the construction of a Hilbert space from the complex vector space of sections of a line bundle over a symplectic $2n$ manifold (M, ω). The line bundle has a Hermitian metric and connection, with its curvature form the given symplectic form ω. Hermitian operators on the Hilbert space then arise as representations of the Lie algebra of smooth real functions on M. To ensure irreducibility for these representations in the case of classical observables it is necessary for the underlying symplectic manifold to have a polarization (Blattner, 1974; Kostant, 1973).

Definition. A *real polarization* of a symplectic $2n$ manifold (M, ω) is a smooth distribution

$$D: M \to TM: m \mapsto D_m$$

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such that (i) D_m is an n-dimensional subspace of T_mM with the property that for all $m \in M$ it is maximally isotropic,

$$D_m \times D_m \subseteq \ker \omega$$

and (ii) D is involutive, its vector fields form a Lie subalgebra. Such a D is called an involutive *Lagrangian* distribution on M. Two real polarizations D^1, D^2 for M are called *transversal* if for all $m \in M$ they admit the decomposition

$$T_mM = D_m{}^1 \oplus D_m{}^2$$

The sensitivity of the quantization procedure to changes of polarization must still be elaborated. Results published so far have concerned real transversal polarizations (Blattner, 1974; Kostant, 1973). Moreover, even when M is the phase space (cotangent bundle) of a physical system there need not be a natural choice of polarization. This is especially apparent for more complicated systems (like, for example, spinning particles in curved space-time) and it has retarded the application of geometric quantization.

We shall see below how polarizations can be manipulated by exploiting their natural links with projective geometry. For definiteness in the constructions we take $M = R^{2n}$, which is the classical phase space of a free particle in R^n or of the n-dimensional harmonic oscillator. Nevertheless, we expect the procedure to be useful also for more general M. In any case, a theorem from projective geometry generates necessary conditions for bases of n-dimensional subspaces of T_mM to span Lagrangian subspaces.

2. PROJECTIVE GEOMETRY AND REAL TRANSVERSAL POLARIZATIONS

Most of the projective geometry that we use can be found in Baer (1952). Yale (1968, p. 204) gives a proof of the equivalence between the "extended affine space" and the "collapsed vector space" views of a projective space. We make use of the latter approach; for any n-dimensional vector space V, the associated $(n-1)$-dimensional projective space $\overline{V}$ is $\{Q | Q \text{ is a one-dimensional subspace of } V\}$. For the case $M = R^{2n}$ we have, for all $m \in M$, $T_mM = R^{2n}$ and so $\overline{T_mM} = \overline{R^{2n}}$ or RP^{2n-1} in more usual notation.

Definition. A *correlation* (autoduality) $\mathscr{C}$ of a projective space $\overline{V}$ is an inclusion-reversing permutation of its proper subspaces. $\mathscr{C}$ is *symplectic* if $\bar{v} \in \mathscr{C}(\bar{v})$ for all $\bar{v} \in \overline{V}$.

Remark. By "inclusion-reversing" we mean that if $\overline{W}$, $\overline{U}$ are subspaces of $\overline{V}$, then $\overline{W} \subseteq \overline{U}$ if and only if $\mathscr{C}(\overline{W}) \supseteq \mathscr{C}(\overline{U})$. Note that dim $\overline{V} > 1$ for this definition.

> *Theorem 1.* A correlation $\mathscr{C}$ of $\overline{V}$ defines a semibilinear form Ω on V such that for each proper subspace W, $\mathscr{C}(\overline{W}) = \{\bar{v} | \Omega(v, w) = 0, \forall w \in W\}$. Conversely the mapping $\mathscr{C}$ defined by this equation is a correlation if and only if Ω is nondegenerate.

Proof. See Yale (1968), p. 260. ∎

> *Theorem 2.* If $\mathscr{C}$ is a symplectic correlation on $\overline{V}$ and Ω is the semibilinear form representing $\mathscr{C}$, then
> (i) Ω is skew symmetric and bilinear.
> (ii) dim $\overline{V}$ is odd and there exists a basis $u_0, u_1, \ldots, u_{n-1}$ of V such that $\Omega(u_{2_i}, u_{2_i+1}) = 1$, but otherwise $\Omega(u_i, u_j) = 0$.

Proof. See Baer (1951), pp. 106–109, also Brauer (1936) for a proof making more use of coordinates. ∎

Remark. Theorems 1 and 2 allow us to define a symplectic 2-form on a vector space V in terms of a symplectic correlation on the associated projective space $\overline{V}$ when we impose the extra condition $d\Omega = 0$.

Definition. Two subspaces $\overline{W}_1$, $\overline{W}_2$ of a projective space $\overline{V}$ are called transversal if and only if $W_1 \oplus W_2 = V$.

Remark. Plainly, if W_1, W_2 are transversal n-dimensional subspaces of a $2n$-dimensional vector space V, then $\overline{W}_1$, $\overline{W}_2$ are also transversal disjoint $(n - 1)$-dimensional hyperplanes in $\overline{V}$, and conversely.

We now have the necessary machinery to enable us to characterize the real transversal Lagrangian distributions on $M = R^{2n}$ in projective geometrical terms. Essentially, the transversal Lagrangian subspaces of $\overline{T_m M}$ are certain $(n - 1)$-dimensional skew hyperplanes of $\overline{T_m M} = RP^{2n-1}$—namely, those that, with respect to the symplectic correlation $\mathscr{C}$ on RP^{2n-1} defining the symplectic 2-form $\omega = dp^a \wedge dq_a$, $(a = 1, \ldots, n)$ on $M = R^{2n}$, have maximally isotropic n-dimensional associates. This is made more precise in a later paper (Campbell, 1978).

Example. $M = R^4$. In this case the real transversal Lagrangian distributions of M are given by those disjoint real projective lines that, with respect to the symplectic correlation on RP^3 defining the symplectic form $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ on M, have maximally isotropic 2-dimensional associates.

Theorem 2 tells us that dim $\bar{V}$ is odd if a symplectic correlation is defined on $\bar{V}$. This is what we would expect since a symplectic 2-form is defined only on V of even dimension.

We may use the extra conditions on a basis for $V = T_m M$ to give us a necessary condition for bases of n-dimensional subspaces of $T_m M$ to span Lagrangian subspaces, and therefore define real polarizations of R^{2n}, $n > 2$. This will be of use in the classification of such polarizations.

Taking $V = T_m M = R^{2n}$, $n \geqslant 2$, Theorem 2 implies the existence of a basis $u_0, u_1, \ldots, u_{2n-1}$ of $T_m M$ such that

$$\left.\begin{aligned} \omega(u_0, u_1) &= 1 \\ \omega(u_2, u_3) &= 1 \\ &\vdots \\ \omega(u_{2n-2}, u_{2n-1}) &= 1 \end{aligned}\right\} n \text{ conditions, but otherwise } \omega(u_i, u_j) = 0$$

In this case we may take $\omega = dp_a \wedge dq^a$ and $u_0, u_1, \ldots, u_{2n-1}$ as

$$\{\partial/\partial p_1, \partial/\partial q_1, \partial/\partial p_2, \partial/\partial q_2, \ldots, \partial/\partial p_n, \partial/\partial q_n\}$$

We can specify a general basis for any n-dimensional subspace D_m of $T_m M$ as a linearly independent set

$$B(u_i) = \{\underbrace{(a_0 u_0 + \cdots + a_{2n-1} u_{2n-1}), \ldots, (t_0 u_0 + \cdots + t_{2n-1} u_{2n-1})}\}$$

then no element is a multiple of another in the set. An n-dimensional subspace D_m is then Lagrangian if

$$\omega\left(k_1\left(\sum_{i=0}^{2n-1} a_i u_i\right) + \cdots + k_n\left(\sum_{i=0}^{2n-1} t_i u_i\right), l_1\left(\sum_{i=0}^{2n-1} a_i u_i\right) + \cdots + l_n\left(\sum_{i=0}^{2n-1} t_i u_i\right)\right)$$

$$= 0 \qquad \text{for all vectors}$$

$$\left.\begin{aligned} w &= k_1\left(\sum_{i=0}^{2n-1} a_i u_i\right) + \cdots + k_n\left(\sum_{i=0}^{2n-1} t_i u_i\right) \\ v &= l_1\left(\sum_{i=0}^{2n-1} a_i u_i\right) + \cdots + l_n\left(\sum_{i=0}^{2n-1} t_i u_i\right) \end{aligned}\right\} \in D_m$$

In fact, for $n \geqslant 2$, as can be seen from the conditions of the theorem,

$$\left\{\frac{u_{2_i} \in D_m \Rightarrow u_{2_i+1} \notin D_m}{u_{2_i+1} \in D_m \Rightarrow u_{2_i} \notin D_m}\right\}, \forall i$$

is a necessary condition for D_m to be Lagrangian. It is not sufficient, however, since $B(u_i) = \{u_0, (u_1 + u_2)\}$, for example, satisfies the condition but does *not* span a Lagrangian subspace. From the set of all such $B(u_i)$ we can choose those spanning Lagrangian subspaces by inspection.

Example. $T_mM = R^4$ has basis u_0, u_1, u_2, u_3 such that

$$\omega(u_0, u_1) = 1$$
$$\omega(u_2, u_3) = 1$$

Otherwise $\omega(u_i, u_j) = 0$, with $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ and u_0, u_1, u_2, u_3 as

$$\{\partial/\partial p_1, \partial/\partial q_1, \partial/\partial p_2, \partial/\partial q_2\}$$

Let

$$\{(a_0u_0 + a_1u_1 + a_2u_2 + a_3u_3), (b_0u_0 + b_1u_1 + b_2u_2 + b_3u_3)\}$$

be a basis for a two-dimensional subspace of T_mM. Then the necessary condition for such bases to span Lagrangian subspaces is

$$\left\{\frac{u_0 \in D_m \Rightarrow u_1 \notin D_m,\ u_2 \in D_m \Rightarrow u_3 \notin D_m}{u_1 \in D_m \Rightarrow u_0 \notin D_m,\ u_3 \in D_m \Rightarrow u_2 \notin D_m}\right\}$$

For example $\{u_0, u_2\}$ and $\{u_1, u_3\}$ are bases for $D_m{}^1$, $D_m{}^2$ satisfying this condition. The results of Theorem 2 also show that they span Lagrangian subspaces. Since $\{u_0, u_1, u_2, u_3\}$ span T_mM, $D_m{}^1 + D_m{}^2 = T_mM$. Also $D_m{}^1 \cap D_m{}^2 = 0$; so $D_m{}^1$, $D_m{}^2$ are transversal.

Remark. For more general systems $M \neq R^{2n}$. However, given a basis for each T_mM, a similar procedure can be followed to define a necessary restriction on the set of all bases for possible Lagrange subspaces. Of course, any manifold if smooth has at least one chart about each of its points and every such chart about $m \in M$ determines a basis for T_mM. It remains to choose one basis for each point in any particular situation. Presumably the choice should be made smoothly, so what we require is a smooth section of the frame bundle, LM. Some manifolds do not possess such sections; those that do are called parallelizable. The only compact 2-manifolds that are parallelizable are the Klein bottle and the torus. The only spheres that are parallelizable are S^1, S^3 and S^7, all of which have odd dimension. We note that, for a $2n$-manifold, the frame bundle is a principal fiber bundle with structure group $Gl(2n; R)$. Moreover, a connection in it will provide horizontal lifts of vector fields on M and hence parallel transport of bases along curves in M.

3. CHANGING POLARIZATIONS

Blattner (1974) has discussed maps between real and transversal polarizations D^1, D^2 of a symplectic manifold M in terms of the corresponding symplectic automorphisms on M. These symplectic automorphisms induce diffeomorphisms on a line bundle above M which is used in the formulation of geometric quantization. If D^1, D^2 are "unitarily related" (Blattner, 1974, p. 152), then to each line bundle diffeomorphism may be intrinsically associated a unitary operator on the appropriate Hilbert space.

It turns out that we can also discuss maps between real transversal polarizations by using projective geometry. In addition it appears that there is an intriguing link between the two approaches.

Definition. A permutation f of $\bar{V}$ is a *projective transformation* if and only if there exists a linear transformation $g\colon V \to V$ such that $f\bar{v} = \bar{g}\bar{v}$ for all $\bar{v} \in \bar{V}$.

> *Theorem 3.* The general linear group G for a vector space V induces a group, the projective group, of transformations of $\bar{V}$.

Proof. See Yale (1968), p. 234. ∎

Remark. Let $V = T_mM = R^{2n}$. Consider transversal Lagrangian subspaces $D_m{}^1$, $D_m{}^2$. The nonsingular maps among them are given by certain projective transformations in RP^{2n-1}—namely, those among the skew $(n-1)$-dimensional hyperplanes in RP^{2n-1} associated with $D_m{}^1$, $D_m{}^2$ by means of the correspondence between the symplectic correlation on RP^{2n-1} and the symplectic form $dp_a \wedge dq^a$ $(a = 1, \ldots, n)$ on M.

4. CONCLUDING REMARKS

It follows from the fundamental theorem of projective geometry that distinct points in RP^1 are related by a unique projective transformation. Accordingly, real transversal polarizations of $M = R^2$ are related by unique linear transformations on T_mM.

For the vector space $V = R^{2n}$ we have the general linear group of its automorphisms, $Gl(2n; R)$. When scalar multiples are factored out we obtain the appropriate projective group $PGl(2n; R)$. This is because a point in the projective space $\bar{V}$ is an equivalence class of points in V under the relation of collinearity. In consequence, maps among real transversal Lagrangian distributions are governed by elements from $PGl(2n; R)$. So with a suitable projective interpretation for the involutive condition we can characterize real transversal polarizations in projective geometry and interpret maps among them via the projective group. Also we have a scheme for finding real transversal pairs of polarizations. For this to be unambiguous in the general case we need the existence of a section of the frame bundle.

It is an intriguing fact for geometric quantization that symplectic structures have also a significant role in projective geometry. Linear transformations leaving a symplectic 2-form invariant correspond to projective transformations commuting with a symplectic correlation. This is discussed to some extent by Dieudonné (1951). By this means we have a projective framework for handling the symplectic automorphisms on a symplectic manifold that arise in changes of real transversal polarization. It remains to be seen what restrictions will be implied for $PGl(2n; R)$ to preserve the required symplectic symmetries. It is an open question whether there exists a projective analog for the geometric quantization procedure of associating a unitary operator with each diffeomorphism on the quantization line bundle induced by symplectic automorphisms of the underlying manifold.

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